Stability of the Equilibrium Position of the Centre of Mass of a System of Inter-connected Satellites in Elliptic Orbit Under the Influence of Perturbative Forces

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Abstract - The proposed topic is to examine the stability of the centre of mass of two satellites connected by extensible cable in elliptic orbit under the influence of air resistance and the shadow of the oblate earth due to solar pressure. We have obtained two equilibrium points out of which one is found to be stable in the sense of Liapunov.

1 INTRODUCTION

THIS paper is devoted to study the stability of equilibrium points of the centre of mass of the system of two satellites connected by light, Flexible and extensible cable under the perturbative forces mentioned in the Abstract in elliptic orbit. First of all we have derived the differential equations of motion by using Lagrange's equations of motion of first kind in linearised and normalized form. The general solution of the system of differential equations are beyond our reach. Hence, we need to use averaged values of the periodic terms present in the differential equations. Then we find Jacobi's integral for the problem and then we get two equilibrium points of the system. Out of these two equilibrium points, one is found to be stable in the sense of Liapunov.

2 EQUATIONS OF MOTION OF THE SYSTEM

The equitation of motion of one of two satellites with respect to the centre of mass of the system moving along Keplerian elliptical orbit in Nechvill's coordinates have been obtained by exploiting Lagrange's equations of motion of first kind in the form :

$$x''-2y'-3x\rho+\frac{4Bx}{\rho}+f\rho\rho^{1}+A\rho^{3}\Psi_1\cos\in\cos(\nu-\alpha)=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_0}{\rho r}\right]x$$

Ψ_1 = Shadow function parameter.

$$y''+2x'-\frac{By}{\rho}-+f\rho^2-A\rho^3\Psi_1\cos\in\sin(\nu-\alpha)=-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_0}{\rho r}\right]y$$

and

$$z''+z'-\frac{Bz}{\rho}-A\rho^3\Psi_1\sin\in \quad =-\overline{\lambda_\alpha}\rho^4\left[1-\frac{l_0}{\rho r}\right]z$$

…………………………(1) ………………(1)

where,

$\rho = \frac{1}{1+e\cos\nu}$; v being true anomaly of the orbit of the centre of mass

$r=\sqrt{x^2+y^2+z^2}$; $\overline{\lambda}_\alpha=\frac{p^3}{\mu}\lambda_\alpha$; l_0 = Natural length of cable connecting the two satellites of mass m_1 and m_2

$A=\frac{P^3}{\mu}\left(\frac{B_1}{m_1}-\frac{B_2}{m_2}\right)$ = Solar pressure parameter

$B=-\frac{3k_2}{P^2}$ = Oblateness parameter.

$\overline{\lambda}_\alpha=\frac{p^3}{\mu}\lambda_\alpha; l$

$f=\frac{a_1p^3}{\sqrt{\mu p}}$ = Air resistance parameter.

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Here dashes denotes the differentiations with respect to true anomaly ν.

The condition of constraint is given by,

$$x^2 + y^2 + z^2 \le \frac{l_0^{\,2}}{\rho^2} \quad \ldots\ldots\ldots\ldots (2)$$

For two dimensional equations of motion, (1) and (2) take the form

$$x''-2y'-3x\rho+\frac{4Bx}{\rho}+f\rho\rho^1+A\rho^3\Psi_1\cos\in\cos(\nu-\alpha)=-\overline{\lambda}_\alpha\rho^4\left[1-\frac{l_0}{\rho r}\right]x$$

and

$$y''+2x'-\frac{By}{\rho}+f\rho^2-A\rho^3\Psi_1\cos\in\sin(\nu-\alpha)=-\overline{\lambda}_\alpha\rho^4\left[1-\frac{l_0}{\rho r}\right]y$$

$$\ldots\ldots\ldots\ldots(3)$$

$$x^2+y^2\le\frac{l_0^{\,2}}{\rho^2} \quad \ldots\ldots\ldots\ldots(4)$$

For averaging the secular terms due to periodic terms present in (3), we have

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\rho}\,d\nu=1$$

$$\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\rho}\,d\nu=\frac{1+e^2}{2}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho\,d\nu=\frac{1}{\sqrt{1-e^2}}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho^2\,d\nu=\frac{1}{(1-e^2)^{\frac{3}{2}}}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho^3\,d\nu=\frac{2+e^2}{2(1-e^2)^{\frac{5}{2}}}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho^4\,d\nu=\frac{2+3e^2}{2(1-e^2)^{\frac{7}{2}}}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\rho\delta^1\,d\nu=0$$

$$\frac{1}{2\pi}\int_0^{2\pi}A\rho^3\psi_1\cos\in\cos(\nu-\alpha)d\nu=$$

$$\frac{1}{2\pi}\left[\int_{\substack{-\theta\\ \Psi_1=0}}^{\theta}A\rho^3\psi_1\cos\in\cos(\nu-\alpha)d\nu+\int_{\substack{\theta\\ \Psi_1=1}}^{2\pi-\theta}A\rho^3\psi_1\cos\in\cos(\nu-\alpha)d\nu\right]$$

$$=\frac{-(2+e^2)A\cos\in\cos\alpha\sin\theta}{2\pi(1-e^2)^{\frac{5}{2}}}$$

$$and\quad \frac{1}{2\pi}\int_0^{2\pi}A\rho^3\psi_1\cos\in\sin(\nu-\alpha)d\nu=$$

$$\frac{1}{2\pi}\left[\int_{\substack{-\theta\\ \Psi_1=0}}^{\theta}A\rho^3\psi_1\cos\in\sin(\nu-\alpha)d\nu+\int_{\substack{\theta\\ \Psi_1=1}}^{2\pi-\theta}A\rho^3\psi_1\cos\in\sin(\nu-\alpha)d\nu\right]$$

$$=\frac{(2+e^2)A\cos\in\sin\alpha\sin\theta}{2\pi(1-e^2)^{\frac{5}{2}}}$$

$$\ldots\ldots\ldots\ldots(5)$$

Where θ is taken to be constant

Using (5) in (3), we get

$$x''-2y'-\frac{3x}{(1-e^2)^{\frac{1}{2}}}+4Bx-\frac{A(2+e^2)\cos\in\cos\alpha\sin\theta}{2\pi(1-e^2)^{\frac{5}{2}}}=-\overline{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}-\frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}}\right]x$$

and

$$\cdots y''+2x'\frac{+f}{(1-e^2)^{\frac{3}{2}}}-BY-\frac{A(2+e^2)\cos\in\sin\alpha\sin\theta}{2\pi(1-e^2)^{\frac{5}{2}}}=-\overline{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}-\frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}}\right]y$$

$$\ldots\ldots\ldots\ldots(6)$$

Where $r=\sqrt{x^2+y^2}$

The condition of constraint given by (4) takes the form

$$x^2+y^2\le\left(1+\frac{e^2}{2}\right)\ell_o \quad \ldots\ldots\ldots\ldots (7)$$

From (6) it follows that the true anomaly v does not appear explicitly in the equations of motion. Hence there must exist Jacobi's integral for the problem.

Multuplying the two euqtions of (6) by 2x′ and 2y′ repectively and adding them together and then after integrating, we get Jacobi's integral in the form.

$$x'^2+y'^2-\frac{3x^2}{(1-e^2)^{\frac{1}{2}}}+4Bx^2-By^2+\frac{2fy}{(1-e^2)^{\frac{3}{2}}}-\frac{(2+e^2)}{\pi(1-e^2)^{\frac{5}{2}}}A\cos\in\sin\theta\{x\cos\alpha+y\sin\alpha\}$$

$$=+\frac{\overline{\lambda}_\alpha(2+3e^2)(x^2+y^2)}{2(1-e^2)^{\frac{7}{2}}}-\frac{\overline{\lambda}_\alpha\ell_o(2+e^2)(x^2+y^2)^{\frac{1}{2}}}{(1-e^2)^{\frac{5}{2}}}$$

$$=he \quad \ldots\ldots\ldots\ldots(8)$$

Where he is the constant of integration.

From (8), We get the curve of zero velocity on putting $x^2 + y'^2 = o$ as

$$\frac{3x^2}{(1-e^2)^{\frac{1}{2}}} - 4Bx^2 + By^2 + \frac{(2+e^2)}{\pi(1-e^2)^{\frac{5}{2}}} A\cos\in\sin\theta\{x\cos\alpha + y\sin\alpha\}$$
$$-\frac{2fy}{(1-e^2)^{\frac{3}{2}}} + \frac{\bar{\lambda}_\alpha \ell_o (2+e^2)(x^2+y^2)^{\frac{1}{2}}}{(1-e^2)^{\frac{5}{2}}} - \frac{\bar{\lambda}_\alpha (2+3e^2)(x^2+y^2)}{2(1-e^2)^{\frac{7}{2}}} + he = o$$

………………(9)

From (9) it follows that the satellite of mass m_1 will be moving within the boundaries of the curve of zero Velocity of (8) for different values of Jacobi's constant he.

EQUILIBRIUM POINT OF THE SYSTEM

The equilibrium positions of the system are obtained by giving constant values of the coordinetes in the equation (6).

Let x= x_o and y = y_o give the equilibrium position where x_o and y_o are constants. Then we have

$$x' = x'' = 0 \quad and \quad y' = y'' = 0$$.

....................................(10) …

…………(10)

Using (10) in (6), we get

$$-\left[\frac{3}{\sqrt{1-e^2}} - 4B\right]x_0 - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A\cos\in\cos\alpha\sin\theta}{\pi}$$
$$= -\bar{\lambda}_\alpha\left(\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right)x_o$$

$$and - By_o + \frac{f}{(1-e^2)^{\frac{3}{2}}}_0 - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} \frac{A\cos\in\sin\alpha\sin\theta}{\pi}$$
$$= -\bar{\lambda}_\alpha\left(\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right)y_o$$

....................................(11)

Where $r_0 = \sqrt{x_0^{\ 2} + y_0^{\ 2}}$

………………(11)

From (11) it follows that it is impossible to find particular solution of the algebraic equations in its present from. Hence, for obtaining equilibrium position, we put $\alpha = 90^0$ in (11) and then we get

$$-\left[\frac{3}{\sqrt{1-e^2}} - 4B\right]x_0 = -\bar{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right]x_o$$

$$and - By_o + \frac{f}{(1-e^2)^{\frac{3}{2}}} - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} A_1 = -\bar{\lambda}_\alpha\left(\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r_o(1-e^2)^{\frac{5}{2}}}\right)y_o$$

When $r_0 = \sqrt{x_0^{\ 2} + y_0^{\ 2}}$ $\quad and \quad A_1 = \dfrac{A\cos\in\sin\theta}{\pi}$

.......................(12)

Solving the two equations of (12), we get two equilibrium positions of the system and are given by (b_1,c_1) and (o,c_2). Out of these two equilibrium points, only (o,c_2) gives the meaningful value of the Hook's modulus of elasticity λ. Thus, we have to test the stability of the equilibrium point (o,c_2) where,

$$[o, c_2] = \left[o\ , \frac{(\bar{\lambda}_\alpha \ell_o + A_1)(2+e^2)(1-e^2) - 2f(1-e^2)^2}{\bar{\lambda}_\alpha (2+3e^2) - 2B(1-e^2)^{\frac{7}{2}}}\right]$$

……………..(13)

Stability of the equilibrium point [o,c_2] given by (13)

Let the equilibrium point be given by,

x = 0 and y = c_2

Let δ_1 and δ_2 be the small variations in x and y conditions of the given equilibrium positions. Then we have

$$x = \delta_1 \qquad and \qquad y = c_2 + \delta_2$$
$$\therefore x' = \delta_1' \qquad ; \quad y' = \delta_2' \quad .... (14)$$
$$x'' = \delta_1'' \qquad ; \quad y'' = \delta_2''$$

Using (14) in (6); we get on putting $\alpha = 90^0$ and $\dfrac{A\cos\in\sin\theta}{\pi} = A_1$ as

$$\delta_1'' - 2\delta_2'\left[\frac{3}{(1-e^2)^{\frac{1}{2}}} - 4B\right]\delta_1 = -\bar{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}}\right]\delta_1$$

and $$\delta_2'' + 2\delta_1' - B(c_2 + \delta_2) - \frac{(2+e^2)}{2(1-e^2)^{\frac{5}{2}}} A_1 + \frac{f}{(1-e^2)^{\frac{3}{2}}}$$

$$= -\bar{\lambda}_\alpha\left[\frac{(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} \frac{-(2+e^2)\ell_o}{2r(1-e^2)^{\frac{5}{2}}}\right](c_2 + \delta_2)$$

Where

r = $\sqrt{\delta_1^{\,2} + (c_2+\delta_2)^{\,2}}$

Multiplying the two equations of (15) by $2\delta_1'$ and 2 $(c_2+\delta_2)^1$ respectively and adding them together. We get after integrating.

$$\delta_1'^2 + \delta_2'^2 - \left[\frac{3}{(1-e^2)^{\frac{1}{2}}} - 4B\right]\delta_1^{\,2} - B(c_2+\delta_2)^2 + \frac{2f(c_2+\delta_2)}{(1-e^2)^{\frac{3}{2}}} - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}}A_1(c_2+\delta_2) + \frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}}\left\{\delta_1^2+(c_2+\delta_2)^2\right\} - \frac{\overline{\lambda}_\alpha(2+e^2)\ell_o}{(1-e^2)^{\frac{5}{2}}}\left\{\delta_1^2+(c_2+\delta_2)^2\right\}^{\frac{1}{2}} = h$$

………………… (17)

where h is the constant of integration.

(17) is known as Jacobi's integral for the variational equations of motion (15)

To examine the stability of the equilibrium point [o. c2] in the sense of Liapunov's, we take Jacobi's integral as Liapunov's function $v(\delta_1', \delta_2', \delta_1, \delta_2)$ and is obtained by expanding the terms of (17) as follows.

$v(\delta_1', \delta_2', \delta_1, \delta_2)$=

$$\delta_1'^2 + \delta_2'^2 + \delta_1^{\,2}\left[\frac{-3}{(1-e^2)^{\frac{1}{2}}} + 4B + \frac{\overline{\lambda}_\alpha(2+3e^2)}{(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda}_\alpha\ell_o(2+e^2)}{2c_2(1-e^2)^{\frac{5}{2}}}\right]$$

$$+\delta_2^{\,2}\left[\frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - B\right]$$

$$+\delta_2\left[\frac{2f}{(1-e^2)^{\frac{3}{2}}} - 2Bc_2 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}}A_1 + \frac{\overline{\lambda}_\alpha(2+3e^2)c_2}{(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda}_\alpha(2+e^2)\ell_o}{(1-e^2)^{\frac{5}{2}}}\right]$$

$$+\left[\frac{2c_2 f}{(1-e^2)^{\frac{3}{2}}} - Bc_2^2 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}}A_1c_2 + \frac{\overline{\lambda}_\alpha(2+3e^2)c_2}{(1-e^2)^{\frac{7}{2}}} + \frac{\overline{\lambda}_\alpha(2+e^2)\ell_o c_2}{(1-e^2)^{\frac{5}{2}}}\right]$$

$$+0(3) = h$$

....................................(18)

Where 0(3) stands for the third and higher order terms in small quantities δ_1 and δ_2.

It follows from Liapunov's theorem an stability that the only criterion for the given equilibrium position (o,c2) given by (13) to be stable is that V defined by (18) must be positive definite and for this the following three conditions must be satisfied :

(i)

$$\frac{2f}{(1-e^2)^{\frac{3}{2}}} - 2Bc_2 - \frac{(2+e^2)}{(1-e^2)^{\frac{5}{2}}}A_1 + \frac{\overline{\lambda}_\alpha(2+3e^2)c_2}{(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda}_\alpha(2+e^2)\ell_o}{(1-e^2)^{\frac{5}{2}}} = 0$$

(ii)

$$4B - \frac{3}{(1-e^2)^{\frac{1}{2}}} + \frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - \frac{\overline{\lambda}_\alpha\ell_o(2+e^2)}{2c_2(1-e^2)^{\frac{5}{2}}} > 0$$

(iii)

$$\frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - B > 0$$

……………………..(19)

Now let us analyze the conditions of (19) for stability of the equilibrium position (o,c2) one by one.

<u>Condition (i) of (19)</u>

L.H.S. of (i) of (19)

$$= c_2\left[\frac{\overline{\lambda}_\alpha(2+3e^2)}{(1-e^2)^{\frac{7}{2}}} - 2B\right] + \frac{2f}{(1-e^2)^{\frac{3}{2}}} - \frac{(2+e^2)A_1}{(1-e^2)^{\frac{5}{2}}} - \frac{\overline{\lambda}_\alpha\ell_o(2+e^2)}{(1-e^2)^{\frac{5}{2}}}$$

$$= 2c_2\left[\frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - B\right] - 2c_2\left[\frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - B\right] = 0$$

Hence first condition by (19) is satisfied.

<u>Condition (iii) of (19)</u>

Since c2 is the y coordinate of the equilibrium point, so its numerator and denominator are both positive.

Hence from (13), we get

$$\frac{\overline{\lambda}_\alpha(2+3e^2)}{2(1-e^2)^{\frac{7}{2}}} - B > 0$$

Hence condition (iii) of (19) is satisfied

L.H.S. of (ii) of (19) =

$$4B - \frac{3}{(1-e^2)^{\frac{1}{2}}} + \frac{\overline{\lambda}_\alpha}{2c_2(1-e^2)^{\frac{7}{2}}}\left[(2+e^2)c_2 - \ell_o(2+e^2)(1-e^2)\right]$$

$$= 4B - \frac{3}{(1-e^2)^{\frac{1}{2}}} + \frac{\bar{\lambda}_\alpha (2+e^2)}{2c_2(1-e^2)^{\frac{7}{2}}}\left[(c_2 - \ell_o) + \ell_o e^2\right]$$

$$= \frac{2c_2\left[4B(1-e^2)^{\frac{7}{2}} - 3(1-e^2)^3\right] + \bar{\lambda}_\alpha (2+e^2)\left[(c_2 - \ell_o) + \ell_o e^2\right]}{2c_2(1-e^2)^{\frac{7}{2}}}$$

> if $c_2 - l_o > o$

Thus, condition (ii) of (19) is satisfied if $c_2 > l_o$

Thus, we find that all the three conditions of (19) for equilibrium point [o.c_2] to be stable are satisfied if $c_2 > l_o$ where c_2 and l_o have their usual meanings.

Therefore, we conclude that the equilibrium point [o, c_2] of the system is stable in the sense of Liapunov if $c_2 > l_o$

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